Безкрайни числови редици
Граница на числова редица

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Редица: $A = a_1, a_2, \ldots, a_n, \ldots$

Една редица се нарича **сходяща**, ако с увеличаване на индекса нейните членове се приближават произволно близко до някаква стойност **L**. Числото **L** се нарича **граница** на редицата **A**.

Граница на редица записваме във вида:

$$\lim_{n\to\infty} a_n = L$$

След определено число **n** всички членове са в околност на **L**.

Безкрайна числова редица се нарича всяка числова функция f, дефинирана в множеството на естествените числа $\mathbf{N}$.

Редицата a_n е **безкрайно малка редица** ако:

$$\lim_{n\to\infty} a_n = 0$$

Редицата a_n е **безкрайно голяма редица** ако:

$$\lim_{n\to\infty} a_n = \infty$$

Да се пресметне границата на редицата a_n:

Задача 1 $$\lim_{n\to\infty}\frac{1}{n}=0$$

$$\frac{1}{б.г.р.}=б.м.р.$$

Задача 2 $$\lim_{n\to\infty}\frac{\sin n}{n}=0$$

$$\frac{огр.\,р.}{б.г.р.}=б.м.р.$$

Задача 3 $$\lim_{n\to\infty}(2n+1)=\infty$$

Задача 4 $$\lim_{n\to\infty}\left(-3+\frac{1}{2n}\right)=-3+\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}=-3$$

Задача 5 $$\lim_{n\to\infty}\left(3.\left(\frac{1}{2}\right)^{n}+1\right)=0+1=1$$

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Задача 6 $\lim\limits_{n\to\infty}\frac{n+5}{n}=\left[\frac{\infty}{\infty}\right]=$

Изнасяме като множител особеността n, за да я съкратим!

$$\lim_{n\to\infty}\frac{\cancel{n}\left(1+\frac{5}{n}\right)}{\cancel{n}}=\lim_{n\to\infty}\left(1+\frac{5}{n}\right)=1+5\lim_{n\to\infty}\frac{1}{n}=1+5.0=1$$

Задача 7 $\lim\limits_{n\to\infty}\frac{5n^2+2}{n}=\lim\limits_{n\to\infty}\frac{\cancel{n}\left(5n+\frac{2}{n}\right)}{\cancel{n}}=\left[\frac{\infty}{\infty}\right]=$

Съкращаваме особеността n

$$=\lim_{n\to\infty}\left(5n+\frac{2}{n}\right)=5\lim_{n\to\infty}n+2\lim_{n\to\infty}\frac{1}{n}=+\infty$$

Задача 8 $\lim\limits_{n\to\infty}\frac{2n^3-4n+3}{5n^3-7n^2}=$

Изнасяме като множител особеността n^3 *, за да я съкратим!*

$$\lim_{n\to\infty}\frac{\cancel{n^3}\left(2-\frac{4}{n^2}+\frac{3}{n^3}\right)}{\cancel{n^3}\left(5-\frac{7}{n}\right)}=$$

Съкращаваме особеността n^3

$$\lim_{n\to\infty}\frac{2-\frac{4}{n^2}+\frac{3}{n^3}}{5-\frac{7}{n}}=\frac{2-0+0}{5-0}=\frac{2}{5}$$

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Задача 9 $$\lim_{n\to\infty}\frac{3n-4}{2n^2+4n-1}=0$$

Задача 10 $$\lim_{n\to\infty}\frac{5n^3-6n}{-3n^2+4}=-\infty$$

Задача 11 $$\lim_{n\to\infty}\frac{2-3n+n^2}{n^2+1}=$$

$$\lim_{n\to\infty}\frac{\cancel{n^2}\left(\frac{2}{n^2}-\frac{3}{n}+1\right)}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)}=\frac{0-0+1}{1+0}=1$$

Задача 12 $\lim\limits_{n\to\infty}\frac{(n+1)(n+2)}{n(n+3)}=\lim\limits_{n\to\infty}\frac{n^2+3n+2}{n^2+3n}=1$

Задача 13

$$\lim_{n\to\infty}\frac{(n+2)!+n!}{(n+2)!-n!}=\lim_{n\to\infty}\frac{n!(n+1)(n+2)+n!}{n!(n+1)(n+2)-n!}=$$

$$\lim_{n\to\infty}\frac{n!\left((n+1)(n+2)+1\right)}{n!\left((n+1)(n+2)-1\right)}=\lim_{n\to\infty}\frac{n^2+3n+3}{n^2+3n+1}=1$$

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Задача 14

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k=\lim_{n\to\infty}\frac{1}{n^3}\left(1+2+\ldots+n\right)=$$

$$\lim_{n\to\infty}\frac{1}{n^3}\frac{n\left(n+1\right)}{2}=\lim_{n\to\infty}\frac{n^2+n}{2n^3}=0$$

Задача 15

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2=\lim_{n\to\infty}\frac{1}{n^3}\left(1^2+2^2+\ldots+n^2\right)=$$

$$\lim_{n\to\infty}\frac{n\left(n+1\right)\left(2n+1\right)}{6n^3}=\lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3}=\frac{1}{3}$$

Граници с q^n

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & \textit{при} \quad |q|<1 \\ 1 & \textit{при} \quad q=1 \\ \nexists & \textit{при} \quad q=-1 \\ +\infty & \textit{при} \quad q>1 \\ \infty & \textit{при} \quad q<-1 \end{cases}$$

Задача 16

$$\lim_{n\to\infty} \frac{2^n + 3^n}{4^n} = \left[\frac{\infty}{\infty}\right] = \lim_{n\to\infty}\left(\left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n\right) = 0$$

Задача 17

$$\lim_{n\to\infty}\frac{3^{n+1}+5^{n+1}}{3^n+5^n}=\left[\frac{\infty}{\infty}\right]=\lim_{n\to\infty}\frac{5^{n+1}\left(\left(\frac{3}{5}\right)^{n+1}+1\right)}{5^n\left(\left(\frac{3}{5}\right)^{n}+1\right)}=5$$

Граници на ирационални функции на n

Задача 18

$$\lim_{n\to\infty}\frac{\sqrt{3n^2+1}}{n+1}=\lim_{n\to\infty}\frac{n\sqrt{3+\frac{1}{n^2}}}{n\left(1+\frac{1}{n}\right)}=\sqrt{3}$$

Задача 19

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+1}=\lim_{n\to\infty}\frac{\sqrt{n}}{n\left(1+\frac{1}{n}\right)}=\lim_{n\to\infty}\frac{1}{\sqrt{n}\left(1+\frac{1}{n}\right)}=0$$

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Задача 20

$$\lim_{n\to\infty}\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)=\lim_{n\to\infty}\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)\frac{\sqrt{n^2+1}+\sqrt{n^2-1}}{\sqrt{n^2+1}+\sqrt{n^2-1}}=$$

$$=\lim_{n\to\infty}\frac{\left(n^2+1\right)-\left(n^2-1\right)}{\sqrt{n^2+1}+\sqrt{n^2-1}}=\lim_{n\to\infty}\frac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}}=\frac{2}{\infty}=0$$

Задача 21

$$\lim_{n\to\infty}\frac{\sqrt{n^2+1}+\sqrt{n}}{\sqrt[4]{n^3+1}-n}=\lim_{n\to\infty}\frac{\cancel{n}\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{\frac{1}{n}}\right)}{\cancel{n}\left(\sqrt[4]{\frac{1}{n}+\frac{1}{n^4}}-1\right)}=\frac{1}{-1}=-1$$

$$\boxed{\lim_{n\to\infty}\sqrt{1+\frac{1}{n^2}}=1}$$

$$\boxed{\lim_{n\to\infty}\sqrt{\frac{1}{n}}=0}$$

$$\boxed{\lim_{n\to\infty}\sqrt[4]{\frac{1}{n}+\frac{1}{n^4}}=0}$$

Намиране на граници с помощта на Неперово число

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \mathrm{e} \approx 2.718\ldots \longrightarrow \text{Неперово число}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+3}\right)^{n+3} = \mathrm{e} \qquad \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2} = \mathrm{e}$$

Задача 22

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+5} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{5} =$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{5} = \mathrm{e}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \mathrm{e}}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{5} = 1^5 = 1}$$

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Задача 23

$$\lim_{n\to\infty}\left(1+\frac{1}{n+15}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{1}{n+15}\right)^{n+15}\left(1+\frac{1}{n+15}\right)^{-15}=\mathrm{e}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n+15}\right)^{n+15}=\mathrm{e}}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n+15}\right)^{-15}=1^{-15}=1}$$

Задача 24

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{n}=\mathrm{e}^{2}$$

Задача 25

$$\lim_{n\to\infty}\left(1+\frac{1}{5n}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{1}{5n}\right)^{5n\frac{1}{5}}=\lim_{n\to\infty}\left(\left(1+\frac{1}{5n}\right)^{5n}\right)^{\frac{1}{5}}=\mathrm{e}^{\frac{1}{5}}$$

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Задача 26

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}=\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n}=\lim_{n\to\infty}\left(\frac{1}{\frac{n}{n-1}}\right)^{n}=$$

$$=\lim_{n\to\infty}\frac{1}{\left(\frac{n}{n-1}\right)^{n}}=\lim_{n\to\infty}\frac{1}{\left(\frac{n-1+1}{n-1}\right)^{n}}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n-1}\right)^{n-1+1}}=$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n-1}\right)^{n-1}}\frac{1}{\left(1+\frac{1}{n-1}\right)^{1}}=\frac{1}{\mathrm{e}}=\mathrm{e}^{-1}$$

Задача 27

$$\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{5}}\right)^{\frac{n}{5}5}=e^{5}$$

Задача 28

$$\lim_{n\to\infty}\left(1-\frac{6}{n}\right)^{n}=e^{-6}$$

от 26 и 27

Задача 29

$$\lim_{n\to\infty}\left(1+\frac{3}{2n}\right)^{n}=e^{\frac{1}{2}3}=e^{\frac{3}{2}}$$

от 25 и 27

Задача 30

$$\lim_{n\to\infty}\left(1-\frac{7}{3n}\right)^{n}=\frac{1}{e^{7\frac{1}{3}}}=e^{-\frac{7}{3}}$$

от 25, 26 и 27

Задача 31

$$\lim_{n\to\infty}\left(1-\frac{3}{2n}\right)^{8n}=\left(e^{-\frac{3}{2}}\right)^{8}=e^{-12}$$

Задача 32

$$\lim_{n\to\infty}\left(\frac{n+3}{n+2}\right)^{n}=\lim_{n\to\infty}\left(\frac{\cancel{n}\left(1+\frac{3}{n}\right)}{\cancel{n}\left(1+\frac{2}{n}\right)}\right)^{n}=\frac{\lim\limits_{n\to\infty}\left(1+\frac{3}{n}\right)^{n}}{\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^{n}}=\frac{e^{3}}{e^{2}}=e$$

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Задача 33

$$\lim_{n\to\infty}\left(\frac{n^2-1}{n^2+5n+6}\right)^{\frac{n}{2}}=\lim_{n\to\infty}\left(\frac{(n-1)(n+1)}{(n+2)(n+3)}\right)^{\frac{n}{2}}=$$

$$=\lim_{n\to\infty}\left(\frac{\cancel{n}^2\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}{\cancel{n}^2\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}\right)^{\frac{n}{2}}=\lim_{n\to\infty}\left(\frac{\cancel{n}^2\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}{\cancel{n}^2\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}\right)^{\frac{n}{2}}=$$

$$=\lim_{n\to\infty}\left(\frac{\cancel{n}^2\left(1-\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)^n}{\cancel{n}^2\left(1+\frac{2}{n}\right)^n\left(1+\frac{3}{n}\right)^n}\right)^{\frac{1}{2}}=\left(\frac{e^{-1}e^{1}}{e^{2}e^{3}}\right)^{\frac{1}{2}}=e^{-\frac{5}{2}}$$

Задача 34

$$\lim_{n\to\infty}\left(1-\frac{1}{5n^2+4n}\right)^{5n^2} = \lim_{n\to\infty}\left(1-\frac{1}{5n^2+4n}\right)^{5n^2+4n-4n} =$$

$$= \frac{\lim_{n\to\infty}\left(1-\frac{1}{5n^2+4n}\right)^{5n^2+4n}}{\lim_{n\to\infty}\left(1-\frac{1}{5n^2+4n}\right)^{4n}} = \mathrm{e}^{-1}\lim_{n\to\infty}\frac{1}{\left(1-\frac{1}{5n^2+4n}\right)^{4n}} =$$

$$= \mathrm{e}^{-1}\lim_{n\to\infty}\left(\frac{5n^2+4n-1}{5n^2+4n}\right)^{-4n} = \mathrm{e}^{-1}\lim_{n\to\infty}\left(\frac{5(n+1)\left(n-\frac{1}{5}\right)}{n(5n+4)}\right)^{-4n} =$$

$$=e^{-1}\lim_{n\to\infty}\frac{\cancel{5^{-4n}}\,\cancel{n^{-4n}}\left(1+\dfrac{1}{n}\right)^{-4n}\cancel{n^{-4n}}\left(1-\dfrac{1}{5n}\right)^{-4n}}{\cancel{n^{-4n}}\,\cancel{5^{-4n}}\,\cancel{n^{-4n}}\left(1+\dfrac{4}{5n}\right)^{-4n}}=$$

$$=e^{-1}\lim_{n\to\infty}\left(\frac{\left(1+\dfrac{1}{n}\right)^{n}\left(1-\dfrac{1}{5n}\right)^{n}}{\left(1+\dfrac{4}{5n}\right)^{n}}\right)^{-4}=e^{-1}\left(\frac{e\,e^{-\frac{1}{5}}}{e^{\frac{4}{5}}}\right)^{-4}=e^{-1}$$

Задача 35

$$\lim_{n\to\infty}\left(\frac{n^3-2n^2-n+2}{n^3-9n}\right)^n=\lim_{n\to\infty}\left(\frac{(n-2)(n+1)(n-1)}{n(n-3)(n+3)}\right)^n=$$

$$=\lim_{n\to\infty}\left(\frac{n\left(1-\frac{2}{n}\right)n\left(1+\frac{1}{n}\right)n\left(1-\frac{1}{n}\right)}{nn\left(1-\frac{3}{n}\right)n\left(1+\frac{3}{n}\right)}\right)^n=\frac{\mathrm{e}^{-2}\,\mathrm{e}^{1}\,\mathrm{e}^{-1}}{\mathrm{e}^{-3}\,\mathrm{e}^{3}}=\mathrm{e}^{-2}$$

Изчисление с Mathematica

Чрез функцията $\mathrm{Limit}\left[f, n \to \mathrm{Infinity}\right]$ *се пресмята символично границата на функцията* f*, когато* ***n*** *клони към безкрайност.*

$\mathrm{Plot}\left[f[x],\{x,a,b\}\right]$ *- рисува графиката на функцията, описана с израза* ***f*** *за независима променлива* ***x*** *в интервала* [a, b].

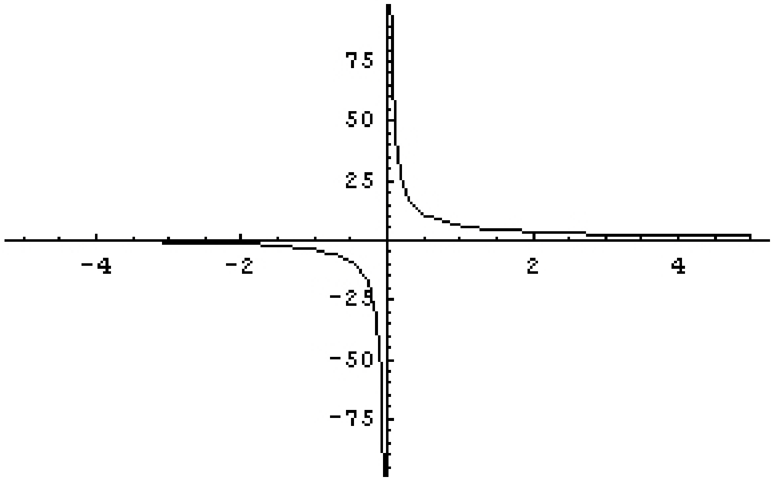
Задача: Да се намери границата на числовата редица с общ член $a_n = \dfrac{n+5}{n}$

$$f := \frac{n+5}{n};$$

```
Print["Границата на функцията при n->∞ e = ", Limit[f, n→Infinity]]
```

```
Границата на функцията при n->∞ e = 1
```

```
Plot[f, {n, -5, 5.}]
```



- Graphics -

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